

Deformed Seiberg-Witten Curves for ADE Quivers

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ABSTRACT: We derive Seiberg-Witten like equations encoding the dynamics of $\mathcal{N} = 2$ ADE quiver gauge theories in presence of a non-trivial Ω -background along a two dimensional plane. The ϵ -deformed prepotential and the chiral correlators of the gauge theory are extracted from difference equations that can be thought as a non-commutative (or quantum) version of the Seiberg-Witten curves for the quiver.

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1. Introduction

It is well-known that the prepotential of a supersymmetric gauge theory with eight supercharges can be computed exactly both at the perturbative and non-perturbative level [1, 2]. These results are elegantly encoded in the geometry of Riemann surfaces. While the original computations were carried out for theories with unitary gauge groups and fundamental or adjoint matter, subsequent works extended those results to account for orthogonal/symplectic groups, quiver (product of groups) gauge theories and symmetric or antisymmetric matter [3]-[9] (see also [10] for a review and references). These extensions were possible once the original results were cast in the language of string and M-theory [11, 12].

There is an alternative way, pioneered by Nekrasov and Okounkov [13], to recover these results from a direct computation in the gauge theory. It is based on the evaluation of the partition function of the theory by means of localization. Before being amenable to such treatment, the partition function Z , which is an integral over the instanton moduli space of the gauge theory of interest, must be suitably deformed by introducing two deformation

parameters ϵ_ℓ which break the Lorentz symmetries and regularize the spacetime volume [14]. Equivalently, one considers the gauge theory on a curved spacetime, the so called Ω -background. For non-trivial ϵ_ℓ , the integral defining $Z(\epsilon_\ell, q)$ localizes around a set of isolated points allowing for its explicit evaluation as a series in the instanton winding number [14]–[20] (see [21] for earlier applications of these ideas). The prepotential \mathcal{F} of the gauge theory is then identified with the free energy of the system in the limit where the ϵ -background is turned off. Sending to zero the parameters ϵ_ℓ , the Seiberg-Witten geometry encoding the non perturbative data emerges from the equations describing the saddle point of the partition function [13].

The study of the gauge theory dynamics beyond the limit $\epsilon_\ell \rightarrow 0$ is also of physical interest. Corrections in ϵ_ℓ to the prepotential describe interactions of gauge theory fields and gravity and summarize an infinite tower of topological string amplitudes [22]. Moreover, the $\mathcal{N} = 2$ partition function $Z(\epsilon_\ell, q)$ at finite ϵ_ℓ has been recently related to conformal blocks of 2D CFT's [23, 24]. At last, the prepotential for the case where one of the ϵ_ℓ is sent to zero has been identified with the Yangian of quantum integrable systems [25]. These results triggered a rapid development of the field along several directions (see [26]–[37] for recent studies of the gauge theory dynamics in the Ω -background).

In [38, 39] it was shown that a saddle point analysis can be adapted to the study of $\mathcal{N} = 2$ $U(N)$ gauge theories in presence of an Ω_ϵ -background with $\epsilon_1 = 0$ and $\epsilon_2 = \epsilon$. The exact prepotential and chiral correlators of the gauge theory in this limit were encoded in a function $y(x)$ solving a Seiberg-Witten like equation. Using a different approach, similar results were also recently obtained in [40, 41].

In this paper, we apply a similar analysis to the case of $\mathcal{N} = 2$ quiver gauge theories. We restrict ourselves to asymptotically free theories with unitary groups and fundamental matter. The quiver gauge theories in this class are given by taking as the quiver diagram an oriented Dynkin diagram of an (Affine or not) ADE Lie algebra. To each node we associate two integers (N_a, n_a) characterizing the ranks of the gauge groups and the number of fundamentals. The arrows in the quiver diagram label bifundamental matter. The study of these quiver gauge theories and the derivation of the corresponding Seiberg-Witten curves governing the dynamics has been recently announced in [42]. The techniques in this paper give an alternative derivation of these results and a generalization to the case of a non-trivial Ω_ϵ -background. Using matrix model techniques the case of quivers in the Ω -background has also been previously studied in [43].

The paper is organized as follows. In section 2 we introduce the gauge theory models. In section 3 we describe the instanton moduli spaces and the saddle point equations determining the leading contribution to the partition function in the limit $\epsilon_1 \rightarrow 0$. The saddle point equations are given by an infinite set of conditions on the finite set (one for each gauge group) of functions $y_a(x)$ encoding the prepotential and chiral correlators of the theory. The set of saddle point equations will be summarized in section 4 as a coupled system of polynomial functional equations for $y_a(x)$ which reduce to the the Seiberg-Witten curves derived in [42] when the ϵ -background is turned off. In section 5 we show how chiral correlators of the gauge theory are computed out of the ϵ -deformed Seiberg-Witten differentials. In section 6, we present alternative ways of writing the ϵ -deformed Seiberg-Witten

equations as a decoupled system of polynomial equations, as a quantum version of Seiberg-Witten curves and as a Thermodynamic Bethe ansatz like integral form. In appendix A we collect some details of the Seiberg-Witten equations for the affine A_1 -quiver gauge theory.

2. Quiver gauge theories

We consider asymptotically free quiver gauge theories with unitary gauge groups and (bi)fundamental matter. A quiver gauge theory in this class is described in terms of a quiver diagram isomorphic to (an orientation of) the Dynkin diagram of an (Affine or not) ADE algebra. To each node “a”, we associate a gauge group $U(N_a)$ and n_a hypermultiplets in the fundamental representation \square_a of $U(N_a)$. An arrow connecting nodes “a” and “b” describes a hypermultiplet in the bifundamental representation $(\square_a, \bar{\square}_b)$ with mass m_{ab} . The hypermultiplet content will be written as

$$\mathcal{H}_{\text{matter}} = \sum_{a,b} c_{ab}(\square_a, \bar{\square}_b) + \sum_a n_a \square_a \quad (2.1)$$

with $c_{ab} = 0, 1$ counting the number of arrows starting at node a and ending on b . We stress the fact that, even if the quiver will be in general described by an orientation of the Dynkin diagram, i.e $c_{ab} \neq c_{ba}$, different orientations of the diagram are physically equivalent since the states in the $\mathcal{N} = 2$ hypermultiplets come always in CPT conjugated pairs. A flip in the orientation of an arrow can indeed, as we will see, be always reabsorbed in a redefinition of the bifundamental masses.

In our study we will limit ourselves to the conformal case since the non-conformal set ups can be obtained from the former by sending some masses to infinity. Conformal invariance translates into the condition

$$\beta_a = -2N_a + (c_{ab} + c_{ba})N_b + n_a = 0 \quad (2.2)$$

which determines the number of fundamentals n_a at each node in terms of the gauge group ranks N_a . For $n_a = 0$, the complete list of conformal quiver gauge theories is given by the affine ADE quivers given in figure 1.

There are several realizations of the quiver gauge theories in string theory. The non-affine A -series is realized in type IIA string theory by suspending D4 branes between a sequence of NS5 branes distributed along a line [11, 12]. The nodes label the intervals between the a^{th} and $(a+1)^{\text{th}}$ NS5 brane. N_a will be the numbers of D4 branes stretched along this interval. The open strings connecting the D4 branes inside the interval realize the $U(N_a)$ gauge degrees of freedom while the open strings stretched between the a^{th} and $(a+1)^{\text{th}}$ or $(a-1)^{\text{th}}$ stacks lead to bifundamental matter. Finally fundamental matter is included by introducing n_a D6 branes. The A -quiver diagram corresponds to the choice $c_{ab} = \delta_{a,b-1}$. Compactifying the brane system on a circle one finds the closed quiver diagram associated to the Dynkin diagram of the Affine \tilde{A} Lie algebra. At strong coupling, the brane system lifts to eleven dimensions and can be described in terms of M5 branes wrapping an A_{r-1} orbifold singularity \mathbb{C}^2/Z_r . Similarly, the D and E series can be described in terms of an M5 brane wrapping the corresponding ADE singularity $\mathbb{C}^2/\Gamma_{ADE}$, with Γ_{ADE}

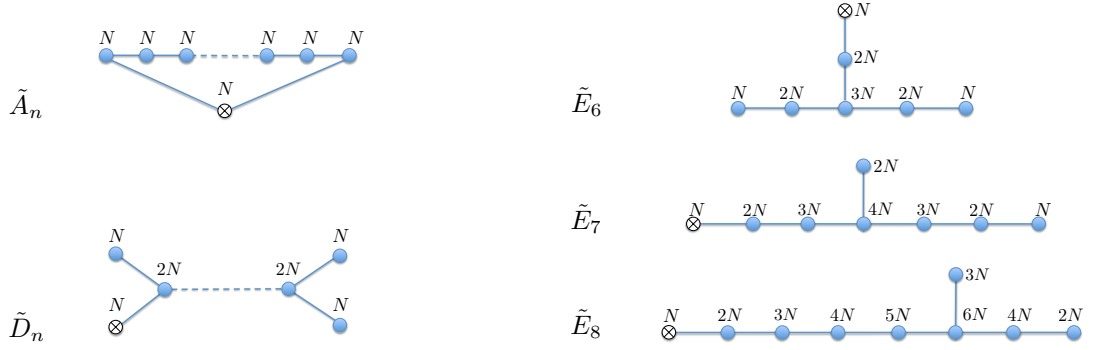


Figure 1: Affine ADE Dynkin diagrams. Non-affine ADE diagrams are obtained by discarding the crossed node. The nodes are labelled by the rank of the corresponding gauge group.

a discrete subgroup of $SU(2)$. Alternatively, the quiver gauge theory can be realized in terms of fractional D3 branes at a $\mathbb{C}^2/\Gamma_{ADE}$ singularity. Fundamental matter in this setting can be included by adding D7 branes.

3. Instanton moduli spaces

In this section we describe the instanton moduli space of the $\mathcal{N} = 2$ quiver gauge theories under discussion. We refer to [16, 20, 44] for a more detailed and a self-contained exposition. We adopt the language of fractional D3-branes (and flavor D7-branes) to describe the quiver gauge theory. In this framework, instantons are viewed as D(-1)-branes. Each node of the quiver corresponds to a type of fractional brane. We consider a general system of k_a , N_a and n_a D(-1), D3 and D7 fractional branes respectively.

Instanton moduli are in correspondence with the massless modes of open strings with at least one end on the D(-1)-branes. The computation of observables in the gauge theory requires the evaluation of integrals over the moduli space spanned by these modes. After a suitable deformation, these integrals localize around a finite set of critical points [14, 15, 16] allowing for their explicit evaluation. Complete localization requires the presence of an Ω background breaking the Lorentz symmetries. Moreover, D-branes have to be distributed along the transverse complex plane with no superposition in such a way that the full symmetry group $\prod_a U(k_a) \times U(N_a) \times U(n_a)$ is broken to its Cartan subgroup. We parametrize by χ_{I_a} , a_{u_a} and m_{s_a} the Cartan elements of $U(k_a)$, $U(N_a)$ and $U(n_a)$ respectively. On the other hand ϵ_1 and ϵ_2 parametrize the Cartan of the Lorentz group. Geometrically, χ_{I_a} , a_{u_a} and m_{s_a} specify the positions of D(-1), D3 and D7 branes respectively along the overall transverse plane. The parameters a_{u_a} , m_{s_a} , ϵ_1 and ϵ_2 are part of the gauge theory data, while χ_{I_a} being an instanton modulus should be integrated over.

It is convenient to encode the symmetry data in the fundamental characters

$$V_a = \sum_{I_a}^{k_a} e^{i\chi_{I_a}} \quad W_a = \sum_{u_a}^{N_a} e^{ia_{u_a}} \quad W_{F,a} = \sum_{s_a=1}^{n_a} e^{im_{s_a}} \quad (3.1)$$

and think of open strings connecting two branes as products of two of these functions. For instance $V_a V_b^*$ represents an open string starting from a D(-1) brane of type “a” and ending on a D(-1) brane of type “b”, $V_a W_b^*$ connects a D(-1) and a D3 brane, and so on. Finally we encode the Lorentz transformation properties of the fields by $T_\ell = e^{i\epsilon_\ell}$.

Localization is based on the existence of a BRST charge \mathcal{Q} . The instanton moduli organize into \mathcal{Q} -pair $(\mathfrak{M}, \mathfrak{N})$ related by

$$\mathcal{Q}\mathfrak{M} = \mathfrak{N} \quad \mathcal{Q}^2\mathfrak{M} = \lambda\mathfrak{M} \quad (3.2)$$

with λ the eigenvalue of the field \mathfrak{M} with respect to the action of an element of the Cartan symmetry group parametrized by $\chi_{I_a}, a_{u_a}, m_{s_a}$ and ϵ_ℓ . The spectrum of eigenvalues λ is summarized in the character

$$\mathbf{T} = \text{tr}_{\mathfrak{M}} e^{i\mathcal{Q}^2} \quad (3.3)$$

In table 1 we display the list of moduli and their contributions to \mathbf{T} .

\mathfrak{M}	\mathbf{T}
B_ℓ, λ_c	$V_a V_a^* (T_1 + T_2) - V_a V_a^* (1 + T_1 T_2)$
w, \bar{w}	$W_a V_a^* + V_a W_a^* T_1 T_2$
$B_{\bar{\ell}}, \lambda_{m'}$	$c_{ab} V_a V_b^* (1 + T_1 T_2) - c_{ab} V_a V_b^* (T_1 + T_2)$
$\mu, \bar{\mu}$	$-c_{ab} W_a V_b^* - c_{ab} V_a W_b^* T_1 T_2$
μ'	$-V_a W_a^*$

Table 1: Instanton moduli for $\mathcal{N} = 2$ quiver gauge theories.

The three main rows display the contributions coming from the moduli associated to gauge, bifundamental and fundamental matter degrees of freedom respectively. The first column displays the highest weight states of the \mathcal{Q} -pairs and the second column their contributions to the character of the given field. In particular the fields $B_{\ell=1,2}, B_{\bar{\ell}=3,4}$, parametrize the positions of the instanton along the spacetime and the transverse space respectively. w, \bar{w} come from open strings stretching between D(-1) and D3 branes. Fermions $\lambda_c, \lambda_{m'}, \mu, \bar{\mu}, \mu'$ contribute with a minus sign and account for the implementation of the ADHM constraints which reduce the number of the degrees of freedom.

3.1 Instanton partition function

The instanton partition function is defined by

$$Z_{\text{inst}} = \int d\mathcal{M} e^{-S_{D(-1)D3}} \quad (3.4)$$

After the equivariant deformation this integral localizes around a set of isolated fixed points of \mathcal{Q}^2 . These points are in one-to-one correspondence with the arrays $Y = \{Y_{u_a}\}$ $u_a = 1, \dots, N_a$ of Young tableaux centered at the D3-brane positions a_{u_a} . Each box in the array of tableaux represents an instanton with position

$$\chi_{I_a}^Y = \chi_{u_a, ij}^Y = a_{u_a} + (i-1)\epsilon_1 + (j-1)\epsilon_2 \quad (3.5)$$

with i, j running over the rows and columns of the tableau Y_{u_a} . The partition function reduces to

$$Z_{\text{inst}} = \sum_Y Z_Y = \sum_Y \frac{q^{|Y|}}{\text{Sdet}_Y \mathcal{Q}^2} \quad (3.6)$$

with $\text{Sdet}_Y \mathcal{Q}^2$ the superdeterminant of \mathcal{Q}^2 evaluated at (3.5) for a given Y . The eigenvalues entering into (3.6) can be extracted from the character \mathbf{T} defined in (3.3) and listed in table 1. Collecting all the contributions one finds

$$\mathbf{T}_Y = \sum_{ab} t_{ab} \mathbf{T}_{ab} - T_F \quad (3.7)$$

with $t_{ab} = \delta_{ab} - c_{ab} e^{im_{ab}}$, and

$$\begin{aligned} \mathbf{T}_{ab, \mathbf{Y}} &= -V_a V_b^* (1 - T_1)(1 - T_2) + W_a V_b^* + V_a W_b^* T_1 T_2 \\ T_{F, Y} &= \sum_a V_a W_{F, a}^* \end{aligned} \quad (3.8)$$

Here the V_a 's are evaluated at the instanton positions (3.5) associated to the critical point described by Y . The extra phases m_{ab} parametrize the masses of bifundamentals.

Collecting the eigenvalues from (3.7, 3.8) and taking $q^{|Y|} = \prod_a [(-)^{N_a - c_{ba} N_b} q_a]^{k_a}$ one finds¹

$$\begin{aligned} Z_Y &= \prod_{a, I_a} \left[-\frac{Q_a(\chi_{I_a}^Y)}{P_a(\chi_{I_a}^Y) P_a(\chi_{I_a}^Y + \epsilon_1 + \epsilon_2)} \prod_{J_a} \Delta(\chi_{I_a}^Y - \chi_{J_a}^Y) \right] \\ &\times \prod_{a, b, I_a} \left[\frac{P_b(\chi_{I_a}^Y - m_{ba})^{c_{ba}} P_b(\chi_{I_a}^Y + \epsilon_1 + \epsilon_2 + m_{ab})^{c_{ab}}}{\prod_{J_b} \Delta(\chi_{I_a}^Y - \chi_{J_b}^Y + m_{ab})^{c_{ab}}} \right] \end{aligned} \quad (3.9)$$

with

$$\begin{aligned} \Delta(x) &= \frac{x(x + \epsilon_1 + \epsilon_2)}{(x + \epsilon_1)(x + \epsilon_2)} & P_b(x) &= \prod_{u_b=1}^{N_b} (x - a_{u_b}) \\ Q_a(x) &= q_a \prod_{s_a=1}^{n_a} (x - m_{s_a}) \end{aligned} \quad (3.10)$$

We notice that a flip in the orientation of an arrow in the quiver diagram sends $c_{ab} \leftrightarrow c_{ba}$ and it can be reabsorbed in the redefinition $m_{ba} \leftrightarrow -m_{ab} - \epsilon_1 - \epsilon_2$. The prepotential of the gauge theory is given by

$$\mathcal{F}_{\text{inst}}(\epsilon_\ell, a, q) = -\epsilon_1 \epsilon_2 \log Z_{\text{inst}}(\epsilon_\ell, a, q) \quad (3.11)$$

The chiral correlators are computed by the localization formula

$$\langle \text{tr}_{N_a} e^{iz\Phi} \rangle = \sum_{u_a=1}^{N_a} e^{iz a_{u_a}} - (1 - T_1^z)(1 - T_2^z) \frac{1}{Z_{\text{inst}}} \sum_Y Z_Y \sum_{I_a} e^{iz \chi_{I_a}} \quad (3.12)$$

¹One can easily check that the number of zero eigenvalues in the numerator and denominator of this formula matches, leading to a finite result.

3.2 Saddle point equations

The instanton partition function is

$$Z_{\text{inst}} = \sum_Y e^{\ln Z_Y} \quad (3.13)$$

with the sum running over the arrays of $\sum_a N_a$ Young tableaux. From (3.9) one finds

$$\ln Z_Y = \sum_{a,b} \sum_{I_a, J_b} F_{ab}(\chi_{I_a}^Y - \chi_{J_b}^Y) + \sum_a \sum_{I_a} M_a(\chi_{I_a}^Y) \quad (3.14)$$

with

$$\begin{aligned} F_{ab}(x) &= \delta_{ab} \ln \Delta(x) - c_{ab} \ln \Delta(x + m_{ab}) \\ M_a(x) &= \ln \left(-\frac{Q_a(x)}{P_a(x)P_a(x + \epsilon_1 + \epsilon_2)} \right) \\ &\quad + \sum_b [c_{ab} \ln P_b(x + m_{ab} + \epsilon_1 + \epsilon_2) + c_{ba} \ln P_b(x - m_{ba})] \end{aligned} \quad (3.15)$$

We are interested in the limit $\epsilon_1 \rightarrow 0$, keeping $\epsilon_2 = \epsilon$ finite. We will follow the strategy in [38, 39] where a similar analysis has been performed for $U(N)$ gauge theories with fundamental and adjoint matter. We refer the reader to these references for further details. In the limit $\epsilon_1 \rightarrow 0$, the instantons of type “a” form a continuous distribution along the intervals

$$\mathcal{I}_a = \cup_{u_a} \mathcal{I}_{u_a} = \cup_{u_a i} [x_{u_a i}^0, x_{u_a i}] \quad (3.16)$$

with

$$x_{u_a, i}^0 = a_{u_a} + (i - 1)\epsilon \quad (3.17)$$

$x_{u_a, i}$ parametrizes the height of the i^{th} columns in the tableau. The details of the Young tableau can be encoded in the instanton density

$$\rho_a(x) = \epsilon_1 \sum_{I_a} \delta(x - \chi_{I_a}) = \begin{cases} 1 & x \in \mathcal{I}_a \\ 0 & x \notin \mathcal{I}_a \end{cases} \quad (3.18)$$

We write

$$\ln Z_Y = \frac{1}{\epsilon_1} \mathcal{H}_{\text{inst}}(\rho) \quad (3.19)$$

with

$$\mathcal{H}_{\text{inst}}(\rho) = \frac{1}{2} \sum_{a,b} \int dx dy \rho_a(x) \rho_b(y) G_{ab}(x - y) + \sum_a \int dx \rho_a(x) M_a(x) \quad (3.20)$$

and

$$G_{ab}(x) = \lim_{\epsilon_1 \rightarrow 0} \frac{1}{\epsilon_1} [F_{ab}(x) + F_{ba}(-x)] \quad (3.21)$$

The partition function becomes

$$Z_{\text{inst}} = \int D\rho e^{\frac{1}{\epsilon_1} \mathcal{H}_{\text{inst}}(\rho)} \quad (3.22)$$

In the limit $\epsilon_1 \rightarrow 0$ the main contribution to the integral comes from the instanton distribution $\rho(x)$ extremizing the Hamiltonian

$$\frac{\delta \rho_a}{\delta x_{uai}} \frac{\delta \mathcal{H}}{\delta \rho_a} = \sum_b \int dy \rho_b(y) G_{ab}(x_{uai} - y) + M_a(x_{uai}) = 0 \quad (3.23)$$

To perform the integral in (3.23) it is convenient to write $G_{ab}(x)$ in the form

$$G_{ab}(x) = \frac{d}{dx} \left[\delta_{ab} \ln \left(\frac{x + \epsilon}{x - \epsilon} \right) + c_{ab} \ln \left(\frac{x + m_{ab}}{x + m_{ab} + \epsilon} \right) + c_{ba} \ln \left(\frac{x - m_{ba} - \epsilon}{x - m_{ba}} \right) \right] \quad (3.24)$$

and introduce the function

$$\mathcal{Y}_a(x) = \prod_{v_a=1}^{N_a} \prod_{i=1}^{\infty} \left(\frac{x - x_{v_a i}}{x - x_{v_a i}^0} \right) \quad (3.25)$$

encoding the information about the x_{uai} . Each term in the saddle point equation (3.23) can be written in terms of this function using the identity

$$\int dy \rho_a(y) \frac{d}{dx} \ln(x - y) = -\ln \mathcal{Y}_a(x) \quad (3.26)$$

The saddle point equations can be thought of as an extremization over the \mathcal{Y} -function. Alternatively, one can use the $x_{ui,a}^0$ -independent combinations²

$$y_a(x) = \frac{\mathcal{Y}_a(x) P_a(x)}{\mathcal{Y}_a(x - \epsilon)} = \prod_{v_a=1}^{N_a} \prod_{i=1}^{\infty} \left(\frac{x - x_{v_a i}}{x - x_{v_a i} - \epsilon} \right) \quad (3.27)$$

We notice that at large x $\mathcal{Y}_a \approx 1$ and

$$y_a \approx x^{N_a} \quad (3.28)$$

In terms of this function the saddle point equation (3.23) can be written in the compact form

$$1 + Q_a(x) \frac{\prod_b y_b(x - m_{ba})^{c_{ba}} y_b(x + m_{ab} + \epsilon)^{c_{ab}}}{y_a(x) y_a(x + \epsilon)} = 0 \quad \text{for } x \in \{x_{uai}\} \quad (3.29)$$

4. Deformed Seiberg-Witten equations

In this section we show that the saddle point equations (3.29) can be equivalently written as Seiberg-Witten like equations for the functions $y_a(x)$. We will follow again the strategy of [38, 39] adapting the analysis to the quiver theory. The main idea is to exploit the saddle point equations to build a set of rational functions, χ_a , of the y 's with no poles in the complex x -plane, i.e. a set of polynomials. In this way, χ_a is completely determined in terms of a finite number of coefficients.

² $y(x)$ is related to the variable $w(x)$ in [39] as $y(x) = 1/w(x)$.

4.1 The $\epsilon = 0$ case

We will start by considering the $\epsilon = 0$ case with massless bifundamental hypermultiplets, i.e. $m_{ab} = 0$, making contact with the results in [42]. In this limit, the Young tableaux profile becomes a smooth curve and the details of the instanton saddle point configuration are encoded in a set of continuous functions $y_a(x)$ with cuts in the x -plane. The Seiberg-Witten equations were deduced [42] from a careful analysis of the discontinuities of these functions across the cuts. We start by illustrating how these results can be recovered from the ϵ -deformed saddle point equations (3.29) after turning off the ϵ -background. The discussion will serve as a warm up for the $\epsilon \neq 0$ analysis in the next subsection.

First we notice that in the limit $\epsilon \rightarrow 0$, the points $x_{u_a i}$ form a continuous distribution around the e_{u_a} 's filling the intervals \mathcal{I}_a defined in (3.16). It is convenient to introduce the following function

$$f_a(x) = y_a(x) + Q_a(x) \frac{\prod_b y_b(x)^{c_{ab} + c_{ba}}}{y_a(x)} \quad (4.1)$$

Using (3.29) one can see that these functions have no poles around \mathcal{I}_a . Indeed, the two terms in the r.h.s. of (4.1) have simple poles in $x_{u_a i}$, but they cancel against each other according to (3.29). On the other hand the functions $f_a(x)$ have poles around \mathcal{I}_b with b linked to a in the quiver diagram. Again these poles can be canceled by the replacement

$$y_b(x) \rightarrow f_b(x) \quad (4.2)$$

in the second term in (4.1). This replacement will however generate new poles at higher orders in the Q_a 's that can be canceled again by the substitution (4.2). According to the case at study, the iteration process will close in a finite or infinite number of steps. Interestingly, one can see that the terms generated in this way match the weights of the basic representations \mathbf{R}_a of the Lie algebra. More in detail, a term of the type $y_1^{p_1} y_2^{p_2} \dots$ can be put in correspondence with a weight (p_1, p_2, \dots) of the representation \mathbf{R}_a . Therefore χ_a can be thought of as a Q -deformed version of its character. The function $\chi_a(y, Q)$ obtained in this way is in general a polynomial in the Q_a 's with coefficients given by ratios of y_a 's. Moreover, $\chi_a(y, Q)$ has no poles in the complex plane and therefore is a polynomial $\mathcal{P}_a(x)$. Using (3.28) one can easily see that all the terms in χ_a grow like x^{N_a} at large x , and therefore \mathcal{P}_a is a polynomial of order N_a in x . One can then write

$$\boxed{\chi_a(y, Q) = \mathcal{P}_a(x)} \quad (4.3)$$

The system of equations (4.3) summarizes the content of the infinite number of algebraic saddle point equations (3.29) in the limit of $\epsilon \rightarrow 0$ and generalizes the Seiberg-Witten equations to the case of a quiver gauge theory. The equations (4.3) can be solved for $y_a(x)$ in terms of the polynomials \mathcal{P}_a . The prepotential and chiral correlators of the quiver gauge theory will be computed in the next section in terms of the periods of the ϵ -deformed Seiberg-Witten differentials built out of the y_a 's.

The result obtained by the iterative procedure (4.2) can be alternatively obtained iterating the maps

$$s_a : y_b(x) \rightarrow \begin{cases} a = b & Q_a(x) y_a(x) \prod_b y_b(x)^{-C_{ab}} \\ a \neq b & y_b(x) \end{cases} \quad (4.4)$$

with

$$C_{ab} = 2\delta_{ab} - c_{ab} - c_{ba} \quad (4.5)$$

the Cartan matrix of the ADE Lie algebra. The map s_a is a sort of Q-deformed Weyl reflection. Indeed, acting with s_a on y_a^{-1} in the denominator brings you back to the previous step in the recursion, i.e. $s_a^2 = 1$. The group W generated by the s_a 's is isomorphic to the Weyl group of the corresponding algebra and was named the iWeyl group in [42]. We refer to the orbit starting from y_a as the iWeyl orbit.

It is important to notice that for some representations of the D and E series higher powers of y_a^n arise in the iWeyl orbit and the replacement $y_a^n \rightarrow f_a^n$ cannot be realized as a result of an iWeyl reflection. In these cases, the extra terms in f_a^n generate new iWeyl orbits completing the weights of the associated representation. For example, for D_4 the **28** leads to a length 24 orbit and four singlets, in the D_5 for the **120** one finds a length 80 and four length **10** orbits, for the **45** we get a 40 plus five singlets and so on and so forth. On the other hand for the A series all characters χ_a are made out of a single orbit.

We write

$$\chi_a(x) = \sum_{w \in \text{Orb}_a} w(y_a) \quad (4.6)$$

with Orb_a the collection of iWeyl orbits spanning all weights of the representation \mathbf{R}_a .

For instance, for A_3 one finds the recursion trees

$$\begin{array}{c}
y_1 \xrightarrow{s_1} \frac{Q_1 y_2}{y_1} \xrightarrow{s_2} \frac{Q_1 Q_2 y_3}{y_2} \xrightarrow{s_3} \frac{Q_1 Q_2 Q_3}{y_3} \\
\\
A_3 : \bigcirc \rightarrow \bigcirc \rightarrow \bigcirc \quad y_2 \xrightarrow{s_2} \frac{Q_2 y_1 y_3}{y_2} \quad \begin{array}{c} s_1 \nearrow \frac{Q_1 Q_2 y_3}{y_1} \searrow s_3 \\ s_3 \searrow \frac{Q_2 Q_3 y_1}{y_3} \nearrow s_1 \end{array} \quad \frac{Q_1 Q_2 Q_3 y_2}{y_1 y_3} \xrightarrow{s_2} \frac{Q_1 Q_2^2 Q_3}{y_2} \\
\\
y_3 \xrightarrow{s_3} \frac{Q_3 y_2}{y_3} \xrightarrow{s_2} \frac{Q_2 Q_3 y_1}{y_2} \xrightarrow{s_1} \frac{Q_1 Q_2 Q_3}{y_1}
\end{array} \quad (4.7)$$

leading to the characters

$$\begin{aligned}
\chi_1 &= y_1 + \frac{Q_1 y_2}{y_1} + \frac{Q_1 Q_2 y_3}{y_2} + \frac{Q_1 Q_2 Q_3}{y_3} \\
\chi_2 &= y_2 + \frac{Q_2 y_1 y_3}{y_2} + \frac{Q_2 Q_3 y_1}{y_3} + \frac{Q_1 Q_2 y_3}{y_1} + \frac{Q_1 Q_2 Q_3 y_2}{y_1 y_3} + \frac{Q_1 Q_2^2 Q_3}{y_2} \\
\chi_3 &= y_3 + \frac{Q_3 y_2}{y_3} + \frac{Q_2 Q_3 y_1}{y_2} + \frac{Q_1 Q_2 Q_3}{y_1}
\end{aligned} \quad (4.8)$$

The Seiberg-Witten curves follow from equating χ_a in (4.8) to \mathcal{P}_a . The same results were recently obtained in [42] by a careful study of the discontinuities of functions y_a crossing the cuts \mathcal{I}_a in the complex plane.

4.2 Turning on the ϵ -background

The analysis in the previous section can be easily adapted to the case of a non-trivial Ω background ϵ and massive bifundamental matter. Using the saddle point equations (3.29)

we can build the function

$$f_a(x) = y_a(x) + Q_a(x - \epsilon) \frac{\prod_b y_b(x - m_{ba} - \epsilon)^{c_{ba}} y_b(x + m_{ab})^{c_{ab}}}{y_a(x - \epsilon)} \quad (4.9)$$

with no poles around \mathcal{I}_a . Indeed, both terms in (4.9) have poles at $\{x_{u_a i} + \epsilon\}$ but they cancel against each other as it follows from (3.29). As before, the function $f_a(x)$ has poles around \mathcal{I}_b that can be canceled by the replacement

$$y_b(z) \rightarrow f_b(z) \quad (4.10)$$

in the numerator of the second term in (4.9). The process is iterated order by order in the Q_i 's and leads, as before, to a continuous function $\chi_a(z)$ with no poles in the complex plane, i.e. a polynomial. Alternatively, one can implement the sequence of replacements (4.10) as a recursion generated at each level by the map

$$y_a(x) \rightarrow Q_a(x - \epsilon) \frac{\prod_b y_b(x - m_{ba} - \epsilon)^{c_{ba}} y_b(x + m_{ab})^{c_{ab}}}{y_a(x - \epsilon)} \quad (4.11)$$

which is a sort of “quantum version” of the iWeyl reflection (4.4). The characters $\chi_a(x)$ are built from $y_a(x)$ following the same steps as before (see (4.7) for example for A_3), with the arguments of the Q_a 's and y_a 's shifted at each step according to (4.11). Explicitly, for an A_2 -quiver with $c_{12} = 1$, $m_{12} = m$, the characters entering in the Seiberg-Witten equations are given by

$$\begin{aligned} A_2: \quad \chi_1 &= y_1(x) + Q_1(x - \epsilon) \frac{y_2(x + m)}{y_1(x - \epsilon)} + \frac{Q_1(x - \epsilon) Q_2(x + m - \epsilon)}{y_2(x + m - \epsilon)} \\ \chi_2 &= y_2(x) + Q_2(x - \epsilon) \frac{y_1(x - m - \epsilon)}{y_2(x - \epsilon)} + \frac{Q_1(x - m - 2\epsilon) Q_2(x - \epsilon)}{y_1(x - m - 2\epsilon)} \end{aligned} \quad (4.12)$$

The first two terms in χ_1 have poles in $x_{u_1 i}$ that cancel against each other according to (3.29). A similar cancellation is achieved between the poles at $x_{u_2 i}$ of the last two terms in χ_1 . The analysis for χ_2 is identical.

The deformed Seiberg-Witten equations can be written as before as

$$\boxed{\chi_a(y, Q) = \mathcal{P}_a(x)} \quad (4.13)$$

with the ϵ -deformed characters $\chi_a(y, Q)$ given in terms of the recursion trees generated by (4.11) starting from $y_a(x)$ and \mathcal{P}_a some polynomials of order N_a .

5. Chiral correlators

Let $y_a(x)$ be a solution of the deformed Seiberg-Witten equations (4.13). The chiral correlators of the $SU(N_a)$ gauge group can be computed according to [39]

$$\langle \text{tr}_{N_a} \Phi^J \rangle = \int_{\gamma_a} \frac{dx}{2\pi i} x^J \partial_x \ln y_a(x) \quad (5.1)$$

with γ_a a cycle around \mathcal{I}_a defined in (3.16). At the instanton level k , this integral receives contributions from the poles of the form $x = e_{u_a} + i\epsilon$ with $i = 0, \dots, k$ and e_{u_a} the zeroes of $\mathcal{P}_a(x)$. More precisely, we write

$$\mathcal{P}_a(x) = g_a \prod_{u_a=1}^{N_a} (x - e_{u_a}) \quad (5.2)$$

with the e_{u_a} 's parametrizing the quantum moduli space of the gauge theory. The coefficient g_a is fixed by matching the leading behavior at large x of the two sides of the equation

$$g_a = x^{-N_a} \lim_{x \rightarrow \infty} \chi_a(x) = \chi_a(Q_a, y_a) \Big|_{Q_a, y_a \rightarrow 1} \quad (5.3)$$

The expression for g_a in the non-conformal case can be found from (5.3) by sending to zero all the q 's associated to a non-conformal node.

The quantum parameters e_{u_a} can be determined in terms of the classical vevs a_{u_a} by inverting the relations

$$a_{u_a} = \int_{\gamma_{u_a}} \frac{dx}{2\pi i} x \partial_x \ln y_a(x) \quad (5.4)$$

with γ_{u_a} a cycle round \mathcal{I}_{u_a} . The non-perturbative prepotential \mathcal{F} of the gauge theory can be extracted from the expression for $\langle \text{tr}_{N_a} \Phi^2 \rangle$ using the Matone relation [45, 46]

$$\langle \text{tr}_{N_a} \Phi^2 \rangle = \sum_{u_a=1}^{N_a} a_{u_a}^2 + 2 q_a \frac{\partial \mathcal{F}}{\partial q_a} \quad (5.5)$$

In the next subsection we will match the results obtained from (5.1) with those coming from the direct evaluation of the instanton partition function (3.12) at $k = 1$ instanton level. We have also performed higher instanton checks of the deformed Seiberg-Witten curves for various quivers of the ADE series.

5.1 $k = 1$

In this section we display the $k = 1$ instanton contribution for a general quiver. The deformed Seiberg-Witten equations up to this order can be written as

$$\mathcal{P}_a(x) = y_a(x) + Q_a(x - \epsilon) \frac{\prod_b y_b(x - \epsilon - m_{ba})^{c_{ba}} y_b(x + m_{ab})^{c_{ab}}}{y_a(x - \epsilon)} + \dots \quad (5.6)$$

The solution of the system of equations (5.6) at first order in the q_a reads

$$y_a(x) = \mathcal{P}_a(x) - Q_a(x - \epsilon) \frac{\prod_b \mathcal{P}_b(x - \epsilon - m_{ba})^{c_{ba}} \mathcal{P}_b(x + m_{ab})^{c_{ab}}}{\mathcal{P}_a(x - \epsilon)} + \dots \quad (5.7)$$

Plugging this into (5.4) one finds

$$\begin{aligned} a_{u_b} &= \sum_{i=0}^1 \text{Res}_{z=e_{u_b}+i\epsilon} x \partial_x \ln y_b(x) + \dots \\ &= e_{u_b} + \frac{1}{P'_b(e_{u_b})} \left[Q_b(e_{u_b}) \frac{\prod_d P_d(e_{u_b} - m_{db})^{c_{db}} P_d(e_{u_b} + \epsilon + m_{bd})^{c_{bd}}}{P_b(e_{u_b} + \epsilon)} \right. \\ &\quad \left. + Q_b(e_{u_b} - \epsilon) \frac{\prod_d P_d(e_{u_b} - \epsilon - m_{db})^{c_{db}} P_d(e_{u_b, b} + m_{bd})^{c_{bd}}}{P_b(e_{u_b} - \epsilon)} \right] + \dots \end{aligned} \quad (5.8)$$

which can be simply inverted and used to evaluate

$$\begin{aligned} \langle \text{tr} \Phi^J \rangle_b &= \sum_{i=0}^1 \sum_{u_b=1}^{N_b} \text{Res}_{z=e_{u_b}+i\epsilon} x^J \partial_x \ln y_b(x) + \dots = \sum_{u_b=1}^{N_b} a_{u_b}^J \\ &+ J \sum_{u_b=1}^{N_b} [(a_{u_b} + \epsilon)^{J-1} - a_{u_b}^{J-1}] Q_b(a_{u_b}) \frac{\prod_d P_d(a_{u_b} - m_{db})^{c_{db}} P_d(a_{u_b} + \epsilon + m_{bd})^{c_{bd}}}{P'_b(a_{u_b}) P_b(a_{u_b} + \epsilon)} + \dots \end{aligned} \quad (5.9)$$

In particular, using the Matone relation (5.5) one finds the first instanton order to the quiver gauge theory prepotential

$$\mathcal{F}_{\text{inst}} = \sum_b \sum_{u_b=1}^{N_b} \epsilon Q_b(a_{u_b}) \frac{\prod_d P_d(a_{u_b} - m_{db})^{c_{db}} P_d(a_{u_b} + \epsilon + m_{bd})^{c_{bd}}}{P'(a_{u_b}) P_b(a_{u_b} + \epsilon)} + \dots \quad (5.10)$$

This formula matches the result for the one-instanton contribution to \mathcal{F}_1 coming from (3.11)

$$\mathcal{F}_1 = -\epsilon_1 \epsilon_2 Z_1 = -\epsilon_1 \epsilon_2 \sum_b \sum_{u_b} Z_{\square_{u_b}} \quad (5.11)$$

with $Z_{\square_{u_b}}$ given by (3.9) with $\chi_{I_b} = a_{u_b}$, $\epsilon_1 = 0$ and $\epsilon_2 = \epsilon$.

6. Alternative formulations of Seiberg-Witten equations

6.1 Seiberg-Witten equations in the non-commutative space

The Seiberg-Witten equations (4.3) for a quiver gauge theory at $\epsilon = 0$ can be alternatively written in terms of the set of polynomial equations [42]

$$\det_{\mathbf{R}_a} (\mathbb{1} - y_a^{-1} \mathbf{g}) \Big|_{\chi_b \rightarrow \mathcal{P}_b} = 0 \quad (6.1)$$

with $\mathbf{g} = \text{diag}\{\chi_{a,i}\}$ a diagonal matrix with entries the components of $\chi_a(x)$. This implies in particular that $\chi_a(x) = \text{tr} \mathbf{g}$. We remark that the system of equations (6.1) is only apparently decoupled since the polynomials appearing in (6.1) depend on all the quantum variables e_{u_a} . The evaluation of the chiral correlators of the $U(N_a)$ gauge theory requires the knowledge of all the periods a_{u_a} (and therefore of all the $y_a(x)$) in order to be able to find the relation between the quantum e_{u_a} and the classical variables a_{u_a} computed from (5.4).

In this section we show how these results can be extended to the $\epsilon \neq 0$ case for the simplest case, the fundamental representation \mathbf{R}_1 of the A_{r-1} quiver³. We will then show how the results can be reinterpreted as a non-commutative (or quantum) version of the Seiberg-Witten curves for the quiver.

By construction

$$0 = \det_{\mathbf{R}_1} (\mathbb{1} - y_1^{-1} \mathbf{g}) = \sum_{j=0}^r \sum_{i_1 < \dots < i_j} \prod_{p=1}^j (-\chi_{1,i_p} y_1^{-1}) \quad (6.2)$$

³We thank R. Poghossian for discussions on this point.

The first equality follows from the fact that $y_1(x)$ is the first eigenvalue of \mathbf{g} . One can easily build an ϵ -deformation of this identity⁴

$$\sum_{j=0}^r \sum_{i_1 < \dots < i_j} \prod_{p=1}^j \left(-\frac{\chi_{1,i_p}(x + p\epsilon - j\epsilon)}{y_1(x - p\epsilon + \epsilon)} \right) = 0 \quad (6.3)$$

which reduces to (6.2) when $\epsilon = 0$. On the other hand, the product in the right hand side of (6.3) can be written in terms of $\chi_j(x)$ using the identity

$$\sum_{i_1 < i_2 < \dots < i_j} \prod_{p=1}^j \chi_{1,i_p}(x - j\epsilon + p\epsilon) = \chi_j(x) \prod_{\ell=1}^j \tilde{Q}_\ell(x - (j - \ell + 1)\epsilon) \quad (6.4)$$

with

$$\tilde{Q}_\ell(x) = \prod_{j=1}^{\ell-1} Q_j(x) \quad \ell = 1, \dots, r \quad (6.5)$$

For $\epsilon = 0$, $Q_a = 1$, the relation (6.4) is nothing but the statement that the node j in the A Dynkin diagram is associated to the antisymmetric product of j fundamentals. For ϵ and Q_a generic the identity (6.4) can be checked using the explicit form of the characters. Replacing $\chi_j(x)$ by $\mathcal{P}_j(x)$ in (6.4) one finds a non-trivial equation that is satisfied precisely at the solutions of the deformed Seiberg-Witten equations $\chi_a(x) = \mathcal{P}_a(x)$, i.e.

$$\boxed{\sum_{j=0}^r (-1)^j \mathcal{P}_j(x) \prod_{\ell=1}^j \frac{\tilde{Q}_\ell(x - (j - \ell + 1)\epsilon)}{y(x - (\ell - 1)\epsilon)} = 0} \quad (6.6)$$

For $\epsilon = 0$, we get the Seiberg-Witten curve

$$W_{A_{r-1}}(x, y) = \sum_{j=0}^r (-1)^j \mathcal{P}_j(x) \prod_{\ell=1}^j (y^{-1} \tilde{Q}_\ell) = 0 \quad (6.7)$$

In the rest of this section we show that the equation (6.6) can be thought as a quantum version of the Seiberg-Witten curve. Indeed (6.6) can be found from (6.7) by promoting the variables x and $z = \log y$ to non-commutative variables. Explicitly, we replace x and z in (6.7) by the operators \hat{x} , \hat{z} satisfying the non-commutativity relation

$$[\hat{z}, \hat{x}] = \epsilon \quad (6.8)$$

The quantum version of the Seiberg-Witten curve can then be written as

$$W_{A_{r-1}}(\hat{x}, \hat{y} = e^{\hat{z}}) |\Psi\rangle = 0 \quad (6.9)$$

with the product over ℓ in (6.7) ordered in such a way that terms with greater ℓ go to the left. Taking $\hat{z} = \epsilon \partial_x$, $|\Psi\rangle = \Psi(x)$ and using $e^{\hat{z}} A(\hat{x}) = A(\hat{x} + \epsilon) e^{\hat{z}}$ we obtain

$$\sum_{j=0}^r (-1)^j \mathcal{P}_j(x) \left(\prod_{\ell=1}^j \tilde{Q}_\ell(x - (j - \ell + 1)\epsilon) \right) \Psi(x - j\epsilon) = 0 \quad (6.10)$$

⁴This relation can be checked recursively by noticing that each term in the sum cancels the previous one leaving a residue canceled by the next, and the process goes on and on and on.

After dividing by $\Psi(x - r\epsilon)$ and defining

$$y(x) = \frac{\Psi(x)}{\Psi(x - \epsilon)} \quad (6.11)$$

we recover the deformed Seiberg-Witten curve (6.6). It would be nice to extend this analysis to the D and E series and other representations of the A series. The main difficulty in doing this comes from the fact that products of representations other than the fundamental leads to products of basic representations that require a more subtle ordering definition. We postpone a more systematic analysis of this issue to future investigations.

6.2 A Thermodynamics Bethe ansatz form

In this section we present an equivalent form of the deformed Seiberg-Witten equations as a set of integral equations of TBA type. We first observe that the saddle point equations (3.29) imply that

$$1 + Q_a(x - \epsilon) \frac{\prod_b y_b(x - m_{ba} - \epsilon)^{c_{ba}} y_b(x + m_{ab})^{c_{ab}}}{y_a(x - \epsilon) y_a(x)} = \frac{\mathcal{P}_a(x)}{y_a(x)} \Theta_a(x) \quad (6.12)$$

with $\Theta_a(x)$ a function with no zeros or poles around \mathcal{I}_a . Indeed the functions in the two sides of (6.12) have zeros at $x_{u_a i} + \epsilon$ and poles at $x_{u_a i}$. Taking the log of the two sides of this equation, multiplying by $\frac{1}{z-x}$ and integrating around \mathcal{I}_a one finds that the $\Theta_a(x)$ -dependent term cancels and one is left with the TBA integral equation

$$\begin{aligned} \log y_a(z) &= \log \mathcal{P}_a(z) \\ &- \int_{\gamma_a} \frac{dx}{2\pi i(x-z)} \log \left(1 + Q_a(x - \epsilon) \frac{\prod_b y_b(x - m_{ba} - \epsilon)^{c_{ba}} y_b(x + m_{ab})^{c_{ab}}}{y_a(x - \epsilon) y_a(x)} \right) \end{aligned} \quad (6.13)$$

This equation can be easily solved order by order in q_a . In particular at one-instanton order one finds

$$y_a(x) = \mathcal{P}_a(x) - Q_a(x - \epsilon) \frac{\prod_b \mathcal{P}_b(x - m_{ba} - \epsilon)^{c_{ba}} \mathcal{P}_b(x + m_{ab})^{c_{ab}}}{\mathcal{P}_a(x - \epsilon)} \quad (6.14)$$

in agreement with the result (5.7) coming from the deformed Seiberg-Witten curve. We remark that function $\Theta_a(x)$ can be determined order by order in Q_a by solving in y_a these equations and requiring that y_a has no poles outside of \mathcal{I}_a .

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A. The affine case

In this appendix we display the deformed Seiberg-Witten curves for the simplest quiver in the Affine series: the A_1 quiver.

A.1 A_1 affine

We consider here an Affine A_1 -quiver gauge theory with $N_1 = N_2 = N$, no fundamental matter $n_1 = n_2 = 0$ and two massless bifundamentals $c_{12} = c_{21} = 1$. Up to $k_1 + k_2 = 3$ one finds

$$\begin{aligned}\chi_1(x) &= y_1(x) + q_1 \frac{y_2(x) y_2(x - \epsilon)}{y_1(x - \epsilon)} + q_1 q_2 \left(y_1(x) + \frac{y_1(x - 2\epsilon) y_2(x)}{y_2(x - 2\epsilon)} \right) + \dots \\ &\quad q_1^2 q_2 \left(\frac{y_2(x) y_2(x - \epsilon)}{y_1(x - \epsilon)} + \frac{y_2(x) y_2(x - 3\epsilon)}{y_1(x - 3\epsilon)} \right) + q_1 q_2^2 \frac{y_1(x) y_1(x - \epsilon) y_1(x - 2\epsilon)}{y_2(x - \epsilon) y_2(x - 2\epsilon)} + \dots \\ \chi_2(x) &= y_2(x) + q_2 \frac{y_1(x) y_1(x - \epsilon)}{y_2(x - \epsilon)} + q_2 q_1 \left(y_2(x) + \frac{y_2(x - 2\epsilon) y_1(x)}{y_1(x - 2\epsilon)} \right) \\ &\quad q_2^2 q_1 \left(\frac{y_1(x) y_1(x - \epsilon)}{y_2(x - \epsilon)} + \frac{y_1(x) y_1(x - 3\epsilon)}{y_2(x - 3\epsilon)} \right) + q_2 q_1^2 \frac{y_2(x) y_2(x - \epsilon) y_2(x - 2\epsilon)}{y_1(x - \epsilon) y_1(x - 2\epsilon)} + \dots\end{aligned}\tag{A.1}$$

The deformed Seiberg-Witten equations are written as

$$\chi_a(y, Q) = \mathcal{P}_a(x)\tag{A.2}$$

with χ_a given by (A.1) and g_a given by the leading term in a large x -expansion of the right hand side of (A.1). One finds

$$g_a = \frac{\sum_{n \in \mathbb{Z}} q^{n^2 - n} q_a^n}{(1 - q) \prod_{n=1}^{\infty} (1 - q^n)}\tag{A.3}$$

with $q = q_1 q_2$.

The $\epsilon = 0$ limit

In the limit $\epsilon = 0$, the right hand side of (A.1) sums up to

$$\begin{aligned}\chi_1(x) &= \frac{y_2(x)}{(1 - q) \prod_{n=1}^{\infty} (1 - q^n)} \left(\frac{q_1}{q_2} \right)^{\frac{1}{4}} \vartheta_2(q_1 \frac{y_2(x)^2}{y_1(x)^2} | q^2) \\ \chi_2(x) &= \frac{y_2(x)}{(1 - q) \prod_{n=1}^{\infty} (1 - q^n)} \vartheta_3(q_1 \frac{y_2(x)^2}{y_1(x)^2} | q^2)\end{aligned}\tag{A.4}$$

with

$$\vartheta_b^a(y|q) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n - \frac{a}{2})^2} y^{n - \frac{a}{2}} e^{-\pi i b(n - \frac{a}{2})}\tag{A.5}$$

and $\vartheta_1 = \vartheta_1^{[1]}$, $\vartheta_2 = \vartheta_2^{[1]}$, $\vartheta_3 = \vartheta_3^{[0]}$ and $\vartheta_4 = \vartheta_4^{[1]}$. Using (A.2) and (A.3), the deformed Seiberg-Witten equations (A.4) can be brought to the form

$$P_{e_1}(x) = y_2(x) \frac{\vartheta_2(q_1 \frac{y_2(x)^2}{y_1(x)^2} | q^2)}{\vartheta_2(q_1 | q^2)}\tag{A.6}$$

$$P_{e_2}(x) = y_2(x) \frac{\vartheta_3(q_1 \frac{y_2(x)^2}{y_1(x)^2} | q^2)}{\vartheta_3(q_1 | q^2)}\tag{A.7}$$

with

$$P_{e_a} = \prod_{u_a=1}^{N_a} (x - e_{u_a}) \quad (\text{A.8})$$

Interestingly, the ratio between these two equations depends only on the combination $y = y_1/y_2$. The function y determines the supergravity profile of the twisted field generated by the system of fractional branes at the A_1 -singularity [44]. Moreover in the limit $q_2 \rightarrow 0$ (where the gauge dynamics of the second node is turned off) (A.6) becomes simply $y_2(x) = P_{e_2}(x)$, while (A.6) leads to

$$y_1^2 - y_1 P_{e_1}(x) (1 + q_1) - q_1 P_{e_2}(x)^2 = 0 \quad (\text{A.9})$$

reproducing the Seiberg-Witten curve for $SU(N)$ gauge theories with $2N$ flavors. The same result follows in the limit $q_1 \rightarrow 0$.

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